A problem on nilpotent maps on $\mathbb{R}^2$

"Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator such that $T^3 = 0$. Determine whether $T^2 = 0$ and justify your answer."

(NUS Linear Algebra AP)

The answer is yes. If $\operatorname{rank} T = 0$, then $T = 0$ and so obviously $T^2 = T^3 = 0$. If $\operatorname{rank} T = 2$, then T is an isomorphism and clearly $T^2, T^3 \neq 0$. Hence it suffices to consider the remaining case $\operatorname{rank} T = 1$. If $T^2 \neq 0$ then $\operatorname{im} T^2$ is a nonzero subspace of the 1-dimensional subspace $\operatorname{im} T$, thus

$$\operatorname{im} T^2 = T(\operatorname{im} T) = \operatorname{im} T.$$

Since T stabilises $\operatorname{im} T$, so does T^2, therefore

$$\operatorname{im} T^3 = T^2(\operatorname{im} T) = \operatorname{im} T \neq \{0\},$$

so $T^3 \neq 0$.

Way Yan
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