Integer harmonic numbers

Theorem. *The only nonnegative integers n for which H_n is an integer are 0 and 1.*

(Knuth, TAOCP Vol. 1, Exercise 1.2.7.19)

Proof. The proof is motivated by the following pattern.

n	2	3	4	5	6	7	8
H_n	$\frac{3}{2\cdot 1}$	$\frac{11}{2\cdot 3}$	$\frac{25}{2^2\cdot 3}$	$\frac{137}{2^2\cdot 15}$	$\frac{99}{2^2\cdot 5}$	$\frac{363}{2^2\cdot 35}$	$\frac{761}{2^3\cdot 35}$

Notice how the numerator of each H_n is even, and the denominator has the form $2^k a$ where a is odd and k is the largest integer satisfying $2^k \leq n$. I define the *twoness* of H_n to be the integer k, and an *n-good number* to be any integer k satisfying $2^k \leq n$. Thus we can say that from the table, the twoness of H_n is the largest n-good number. Note that the twoness can also be computed using a non-reduced fractional representation: given any representation p/q of H_n where p is odd, we have $q = 2^k a$ where a is odd and k is the twoness of H_n.

I proceed via induction from $n = 2$ on the statement 'H_n can be written in the form $p/2^k a$ where p is odd, a is odd and the integer k (which is the twoness from the previous paragraph) is the largest n-good number'. Since the twoness is always ≥ 1, the denominator $2^k a$ is even and so H_n cannot be an integer for all $n \geq 2$.

Start at the base case $n = 2$. The reduced form $H_2 = 3/2$ has odd numerator while its twoness, 1, is the largest 2-good number. Now suppose $H_n = p/2^k a$ where p is odd, a is odd, and the twoness k is the largest n-good number. If $n+1$ is odd then we have

$$H_{n+1} = \frac{p}{2^k a} + \frac{1}{n+1} = \frac{p(n+1) + 2^k a}{2^k a(n+1)}.$$

The numerator is odd since one term is odd and one term is even, $a(n+1)$ is odd, and the new twoness k is still the largest $(n+1)$-good number. On the other hand, if $n+1$ is even then write $n+1 = 2^l b$ where b is odd.

Case 1: $b = 1$. We have $l > k$ since $2^l = n + 1 > n$ implying l is not n-good. Thus

$$H_{n+1} = \frac{p}{2^k a} + \frac{1}{2^l} = \frac{2^{l-k} p + a}{2^l a}.$$

The numerator is odd, a is odd, and the new twoness l is clearly the largest integer satisfying $2^l \leq n+1 = 2^l$.

Case 2: $b \geq 3$. We have $n + 1 > 2^l \cdot 2 = 2^{l+1}$ implying $n > 2^{l+1}$. Thus $l + 1 \leq k$ since $l + 1$ is n-good. Hence

$$H_{n+1} = \frac{p}{2^k a} + \frac{1}{2^l b} = \frac{bp + 2^{k-l} a}{2^k ab}.$$

The numerator is odd, ab is odd, and the new twoness k is still the largest $(n+1)$-good number, since a new power of 2 hasn't been reached yet. This completes the induction. ∎