Fibonacci game

"Two players compete in the following game: There is a pile containing n chips; the first player removes any number of chips except that he cannot take the whole pile. From then on, the players alternate moves, each person removing one or more chips but not more than twice as many chips as the preceding player has taken. The player who removes the last chips wins. (For example, suppose that $n = 11$; player A removes 3 chips; player B may remove up to 6 chips, and he takes 1. There remains 7 chips; player A may take 1 or 2 chips, and he takes 2; player B may remove up to 4, and he picks up 1. There remain 4 chips; player A now takes 1; player B must take at least one chip and player A wins in the following turn.)

What is the best move for the first player to make if there are initially 1000 chips?"

(Knuth, TAOCP Vol. 1, Exercise 1.2.8.37)

I will first solve the problem in the general case then solve it for 1000 chips. First, I will introduce some terminology.

1. The first and second players will be called A and B respectively.
2. An *n-game* will refer to a game starting with n chips. An *F-game* will refer to a game starting with a Fibonacci number of chips.
3. The term *move* can refer to one of two things:
   a. The set of chips removed by a player in his turn ('A's move spans two subgames')
   b. The number of chips removed in the turn ('A's move is guaranteed to be $\leq F_{k-1}$')

The big idea behind the solution is that each n-game can be decomposed into F-subgames. First, I show by induction that every F-game is losing for A by decomposing it into two F-subgames. Then, I use this fact to show that every non-F-game is winning for A, by decomposing it into F-subgames according to Zeckendorf's theorem.

Proposition. *Let F_k denote the kth Fibonacci number (so $F_1 = 1$, $F_2 = 1$ and so on). Then except for $k = 1, 2$, the F_k-game is losing for A, and in fact there is a winning strategy for B such that the last move is always $\leq F_{k-1}$.*

Proof. The proof is by induction. The base cases $k = 3, 4$ can be manually checked. For instance if $k = 3$, then A can only move 1, and afterwards B can win by moving 1. Note that the winning move 1 is $\leq F_2$. Similar reasoning applies for $k = 4$.

Now suppose $k \geq 5$. Since $F_k = F_{k-1} + F_{k-2}$, the F_k-game can be split into two subgames: an F_{k-2}-subgame followed by an F_{k-1}-subgame, as shown below.

$$F_k : \quad \underbrace{\circ\circ\cdots\circ\circ}_{F_{k-2}} \quad \underbrace{\circ\circ\circ\circ\cdots\circ\circ\circ\circ}_{F_{k-1}}$$

For the first move, A can make any move m between 1 and $F_k - 1$ (inclusive). Divide m into two cases:

1. If $m \geq F_{k-2}$, then B can win by moving $F_k - m$. Note that the winning move $F_k - m$ satisfies $F_k - m \leq F_k - F_{k-2} = F_{k-1}$.
2. If $m < F_{k-2}$, then we can pretend that A and B are currently playing the F_{k-2}-subgame. By induction, B has a winning strategy for the F_{k-2}-subgame such that the last move is $\leq F_{k-3}$. After this last move, A is once again first to move in the following F_{k-1}-subgame. Due to the previous winning move, A's maximum move is $2F_{k-3}$ by the game rules. Since $2F_{k-3} \leq F_{k-2} + F_{k-3} = F_{k-1}$, A cannot immediately win the second subgame by moving the remaining chips and is forced to play it normally. Once again, by induction B has a winning strategy such that the last move is $\leq F_{k-2}$. ∎

Theorem. *If n is not Fibonacci, then the n-game is winning for A.*

Proof. By Zeckendorf's theorem, n can be written as a sum

$$F_{k_1} + F_{k_2} + \ldots + F_{k_m}$$

such that $k_i \geq k_{i+1} + 2$ and $k_m \geq 2$. Thus the n-game can be split into m F-subgames as follows:

$$n: \quad \underbrace{\circ \cdots \circ}_{F_{k_m}} \quad \underbrace{\circ\circ \cdots \circ\circ}_{F_{k_{m-1}}} \quad \cdots \quad \underbrace{\circ\circ\circ\circ \cdots \circ\circ\circ\circ}_{F_{k_1}}$$

A's winning strategy is to first move F_{k_m}. This immediately ends the first subgame, and now B is first to move in the following $F_{k_{m-1}}$-subgame. B's maximum move is $2F_{k_m}$, and since

$$2F_{k_m} < F_{k_m} + F_{k_m+1} = F_{k_m+2} \leq F_{k_{m-1}},$$

B cannot bypass this subgame and must play it normally. By the previous proposition, A has a winning strategy (since A moves second!) such that the last move is $\leq F_{k_{m-1}-1}$. Once again, B is forced to start first in the following $F_{k_{m-2}}$-subgame and cannot bypass it, and A has a winning strategy, and so on. Eventually A will win the last subgame, meaning it wins the entire n-game. ∎

Now to answer the original question: the Zeckendorf representation of 1000 is $987 + 13$, thus A should move 13.

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